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A New Approach to the Numerical Solution of Integral Equations*

IAN H. SLOAN, B. J. BURN, AND N. DATYNER

Department of Applied Mathematics, University of New South Wales, Kensington, New South Wales 2033, Australia

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A new degenerate-kernel approach is developed for the numerical solution of Fredholm integral equations of the second kind, $y = f + \lambda Ky$. An essential feature is that the rank-N approximate kernel is constructed to be exact when it operates in a certain N-dimensional subspace, the subspace being chosen for its suitability for approximating y. The simplest version of the method is equivalent to a single iteration of the method of moments, or Galerkin method, and is very similar to the method of moments in its computational requirements, but nevertheless is not at all similar in its performance. Numerical examples, including one with a logarithmic singularity in the kernel, show that the simplest version gives consistently better results than the method of moments, the errors typically being smaller by one or more factors of ten. A somewhat more elaborate version of the method is found to give results that are at least marginally better again.

1. INTRODUCTION

In this paper we develop a new degenerate-kernel approach to the numerical solution of Fredholm integral equations of the second kind,

$$y(t) = f(t) + \lambda \int_a^b K(t, s)\, y(s)\, ds, \qquad a \leqslant t \leqslant b, \tag{1}$$

or

$$y = f + \lambda Ky, \tag{2}$$

where y and f are real- or complex-valued functions in $L^2(a, b)$, and K is a square-integrable kernel [1].

Since equations of this form arise in many areas, the present discussion is not tied to a specific physical context. The methods were actually conceived, however, in relation to nuclear few-body collision problems. (In that area there is a particular

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interest in degenerate-kernel methods, because of the great simplifications which they often allow.) Specific applications and extensions to few-body collision problems will be described separately [2, 3].

The vital step in any degenerate-kernel method [4, 5] for Eq. (2) is the construction of the degenerate kernel K_N, after which the approximate equation

$$y_N = f + \lambda K_N y_N$$

can easily be solved by reduction to algebraic linear equations. The distinctive feature of the present approach is that we try to make K_N a good approximation *in the context in which K actually occurs in the integral equation*, i.e., as an operator on the solution y. The standard methods [4, 5], in contrast, pay no special attention to the solution, but rather attempt to find an approximate kernel that is a good approximation over the whole space.

The way in which $K_N y$ is constrained to be a good approximation to Ky is by constructing K_N to have the property

$$K_N u_n = K u_n, \qquad n = 1,..., N, \tag{3}$$

where the $\{u_n\}$ are a linearly independent set of functions, chosen for their suitability as a basis set for approximating y. Then K_N has exactly the same effect as K when it operates on any function in the subspace U_N spanned by $u_1,..., u_N$; and if this subspace contains a good approximation to y, then $K_N y$ will be a good approximation to Ky. (To carry it to an extreme, if we are clever enough to choose the $\{u_n\}$ so that y is actually contained in U_N, then $K_N y$ obviously becomes exact.)

The benefit derived from Eq. (3) can also be expressed in another way: Let us denote by P_N the orthogonal projection operator into the subspace U_N spanned by $u_1,..., u_N$, so that $P_N y$ is the part of the exact solution that lies in this subspace. Then it follows from Eq. (3) that

$$K_N P_N = K P_N,$$

and hence that

$$(K - K_N) y = (K - K_N)(y - P_N y). \tag{4}$$

A conventional error analysis for degenerate-kernel methods makes use of the inequality

$$\|(K - K_N) y\| \leqslant \| K - K_N \| \, \| y \|$$

(where $\| \cdot \|$ denotes any suitable vector norm and consistent operator norm), whereas Eq. (4) leads to the much stronger inequality

$$\|(K - K_N) y\| \leqslant \| K - K_N \| \, \| y - P_N y \|.$$

(In particular, if the set $\{u_n\}$ is complete and the norm is the L^2 norm, then $\| y - P_N y \|$ converges to zero as $N \to \infty$ [1], whereas the corresponding factor in the first inequality is a constant.) A proper error analysis is deferred, however, to a later paper, the emphasis for the present being rather on establishing that the methods are useful and reliable in practice.

The explicit expression for K_N is given in the following section. It will be seen that the expression is only partly determined by Eq. (3), so that the approach yields a family of methods, rather than a single one. Two particular methods are discussed in detail in Sections 3 and 4, and then investigated numerically in Section 6. The simplest of the methods (method 1) is very similar in its computational requirements to the method of moments [4, 5], or Galerkin method, in which an approximate solution is sought in the form

$$y \simeq \sum_{n=1}^{N} a_n u_n ,$$

but the performance will be seen to be very different indeed. An interesting formal relation exists between method 1 and the method of moments, namely that method 1 is formally equivalent to a single iteration of the method of moments. It has been noticed previously [6] that the accuracy of Galerkin methods is often improved by such an iteration.

Readers interested only in computational aspects will need to read only the next section, where the present methods are defined, and Section 6, where numerical examples are discussed.

2. Definition of Methods

As explained above, we introduce a linearly independent set of functions $u_1, \ldots, u_N$ in $L^2(a, b)$, these to be chosen for their suitability as a basis set for approximating y. We also need a second linearly independent set $v_1, \ldots, v_N$ in $L^2(a, b)$, which will be specified later.

We define the approximate kernel of rank N to be

$$K_N(t, s) = \sum_{n,m=1}^{N} K u_n(t)\, D_{nm} \overline{v_m(s)}, \tag{5}$$

where the coefficients D_{nm} are obtained by inverting an $N \times N$ matrix,

$$\begin{aligned} (D^{-1})_{mn} &= \int_a^b \overline{v_m(s)}\, u_n(s)\, ds \\ &= (v_m, u_n), \qquad m, n = 1, \ldots, N, \end{aligned} \tag{6}$$

and where

$$Ku_n(t) = \int_a^b K(t, s)\, u_n(s)\, ds. \tag{7}$$

The requirement that the matrix (6) be nonsingular imposes a constraint on the choice of the $\{v_n\}$. A convenient inner product has been introduced in Eq. (6).

It is easily verified that the K_N so constructed satisfies the desired condition,

$$K_N u_n = Ku_n\,, \qquad n = 1,\ldots, N,$$

whatever the choice of the $\{v_m\}$. However, the choice of the $\{v_m\}$ is not unimportant, as we shall see later.

The solution of the integral equation (2) is then approximated by the solution of the approximate equation

$$y_N = f + \lambda K_N y_N\,.$$

Since this equation has a degenerate kernel, it can easily be solved [4] to give

$$y_N = f + \lambda \sum_{n=1}^{N} a_n K u_n\,, \tag{8}$$

where the coefficients a_n satisfy the N linear equations

$$\sum_{n=1}^{N} [(v_m\,, u_n) - \lambda(v_m\,, Ku_n)]\, a_n = (v_m\,, f), \qquad m = 1,\ldots, N. \tag{9}$$

There remains the question of the choice of the $\{v_m\}$. In this paper we consider in detail only the two choices that seem to us the most attractive. The first is the choice (method 1)

$$v_m = u_m\,, \qquad m = 1,\ldots, N, \tag{10}$$

which has the merit of simplicity, and the second is the choice (method 2)

$$v_m(s) = \int_a^b u_m(t)\, \overline{K(t, s)}\, dt, \qquad m = 1,\ldots, N, \tag{11}$$

which we shall argue is preferable on theoretical grounds, and which turns out in Section 6 to give the best results.

Method 1, and the interesting relation it has to the method of moments, are discussed in the following section, and method 2 is discussed in Section 4. For the purpose of applying the methods, however, the only equations needed are those given above, in particular Eqs. (8)–(11). Method 1 requires the evaluation

of the integrals in (7), and then the inner products in (9). Method 2 is somewhat more difficult, in that it also requires the evaluation of the integrals in (11). However, if K is symmetric these are the same as the integrals Ku_n already evaluated for use in (9), so that the additional difficulty in this case lies only in evaluating what are likely to be more difficult inner products in (9).

3. Discussion of Method 1

Method 1 uses the simple choice

$$v_m = u_m, \qquad m = 1,..., N,$$

for the set $\{v_m\}$. This choice automatically makes the matrix in (6) nonsingular, because its determinant is a Gram determinant [7, p. 59].

This method has an interesting relation with the method of moments [7, 5], or Galerkin method. In that method one seeks an approximate solution of the form

$$y_N^{\text{mom}} = \sum_{n=1}^{N} a_n u_n, \tag{12}$$

and obtains the coefficients a_n by requiring that the approximate solution should satisfy the N equations

$$(u_m, [y_N^{\text{mom}} - f - \lambda K y_N^{\text{mom}}]) = 0, \qquad m = 1,..., N, \tag{13}$$

i.e., by requiring that certain "moments" of the equation be satisfied. It is easy to see that the resulting linear equations are identical to (9), with v_m equal to u_m, hence the coefficients a_n in the moments solution (12) are exactly the same as for method 1. However, the approximate solutions y_N^{mom} and y_N are *not* the same, because the forms of the solutions in the two cases are different. In fact, from (8) and (12) the two approximate solutions are seen to be related by

$$y_N = f + \lambda K y_N^{\text{mom}}, \tag{14}$$

i.e. they are related by a single iteration of the integral equation.

To allow us to better understand the relation between method 1 and the method of moments, it is useful to introduce P_N, the orthogonal projection operator onto the subspace U_N spanned by $u_1,..., u_N$. From Eq. (13), the moments solution satisfies

$$y_N^{\text{mom}} = P_N f + \lambda P_N K y_N^{\text{mom}}, \tag{15}$$

where we have used

$$P_N y_N^{\text{mom}} = y_N^{\text{mom}}.$$

Then on comparing (14) and (15) we immediately conclude

$$P_N y_N = y_N^{\text{mom}}.$$

Thus, if the exact solution y is decomposed by $y = P_N y + (1 - P_N) y$ into a part contained in U_N and another part orthogonal to it, we see that *method* 1 *and the method of moments give exactly the same approximation for* $P_N y$, the part of y contained in U_N.

On the other hand for $(1 - P_N) y$, the part of y orthogonal to U_N, the method of moments obviously predicts zero, whereas method 1 generally gives a nonzero prediction. According to (14), we can reasonably expect this prediction to be a useful one (i.e. to be a better prediction than zero), if the moments solution is a good approximation to y, because the right-hand side is then a good approximation to the right-hand side of the original equation. (If the moments solution is not a good approximation to y then *neither* method can be expected to be useful. The remarks in this paragraph are of course only qualitative. A full error analysis will be given in a future paper.)

It is well known that the method of moments can be expressed as a degenerate-kernel method, indeed (15) is already an integral equation with a degenerate kernel, namely $P_N K$. On the other hand, it is easy to see that the degenerate kernel (5) for method 1 is just $K_N = K P_N$. The formal difference may seem slight, but as we shall see from the numerical results in Section 6, it can be of great importance in practice. Before concluding this part, it is worth mentioning the alternative version [4, p. 151] of the method of moments that begins with

$$y_N^{\text{mom}} = f + \sum_{n=1}^{N} b_n u_n,$$

rather than (12). From our point of view, the important observation here is the easily proved result that the two methods of moments become identical if f is a linear combination of $u_1, \ldots, u_N$, i.e. if $P_N f = f$. This is the case for both of the numerical examples considered in Section 6.

4. Discussion of Method 2

For this case the $\{v_m\}$ are defined by Eq. (11), or concisely

$$v_m = K^* u_m, \qquad m = 1, \ldots, N, \tag{16}$$

where K^* is the adjoint of K, $K^*(s, t) = \overline{K(t, s)}$. For method 2 to be applicable it is necessary that the set $\{K^*u_m\}$ be linearly independent, and that the matrix in (6) be nonsingular. For the latter, it is for example sufficient [7, p. 57], but certainly not necessary, that K be symmetric and positive definite.

With the choice (16) it is easy to verify that $K_N{}^*$ satisfies

$$K_N{}^*u_m = K^*u_m\,, \qquad m = 1,..., N, \tag{17}$$

so that not only K_N but also $K_N{}^*$ is an exact operator in the subspace U_N spanned by $u_1,..., u_N$.

The property (17) can also be expressed in another way: With the aid of the projection operator P_N, Eq. (17) is equivalent to

$$P_N K_N = P_N K, \tag{18}$$

whereas Eq. (3), which of course also holds, is equivalent to

$$K_N P_N = K P_N\,. \tag{19}$$

The first of these properties is shared with the method of moments, and the second is shared with method 1, so that in a sense method 2 has the advantages of both of the others.

In the introduction we stated that our fundamental aim in constructing K_N is to make K_N a good approximation to K in the context in which K actually occurs in the integral equation, i.e. as an operator on y. Taken on its own, the property (19) (which is common to any kernel of the form (5)) leads to an expression for $Ky - K_N y$,

$$Ky - K_N y = (K - K_N)(y - P_N y),$$

which we certainly expect to be small if the basis set $\{u_n\}$ is well chosen, since a good basis set implies that $P_N y \simeq y$. Method 2 goes further than this, since the property (18) also gives an additional constraint on $Ky - K_N y$, namely

$$P_N(Ky - K_N y) = 0, \tag{20}$$

i.e., the projection of $Ky - K_N y$ into the subspace U_N vanishes. This is a very useful constraint in the context of the integral equation, since Ky itself will generally lie almost completely within U_N, if the basis set is well chosen. (This is because Ky is proportional to $y - f$. If the basis set is well chosen for approximating y, in practice it will almost inevitably also be a good basis set for $y - f$.)

5. Other Methods

Of the infinity of other possible choices of the $\{v_m\}$, we mention explicitly only one more, namely the choice

$$v_m(s) = \overline{K(t_m, s)}, \qquad m = 1,\ldots, N, \tag{21}$$

for some suitable set of points $a \leqslant t_1 < t_2 < \cdots < t_N \leqslant b$. This choice leads to the property

$$K_N(t_m, s) = K(t_m, s), \qquad m = 1,\ldots, N,$$

a property analogous in a sense to (18), but with perhaps less clear advantages. From a practical point of view, this method has the disadvantage that one must choose (for each N) the set of points $\{t_m\}$, as well as the functions $\{u_m\}$.

We have in fact carried out numerical calculations with this method, for the first of the examples discussed in the next section, and with the points chosen to be

$$t_m = (2m - 1)/2N, \qquad m = 1,\ldots, N.$$

The numerical results (not shown in this paper) were always somewhat worse than for either method 1 or 2, though usually not greatly so, except for values of λ near a characteristic value, where the differences became considerable.

Bateman's approximation [8] to $K(t, s)$, namely

$$\sum_{n,m=1}^{N} K(t, t_n)\, E_{nm} K(t_m, s),$$

with

$$(E^{-1})_{mn} = K(t_m, t_n), \qquad m, n = 1,\ldots, N,$$

can be obtained formally from (5) by using the above choice (21) for the $\{v_n\}$, and by taking the $\{u_n\}$ to be delta functions, $u_n(t) = \delta(t - t_n)$. The Bateman kernel is exact when either t or s coincides with one of the points t_m. This property is formally analogous to the properties (18) and (19) of method 2, but the Bateman method is expected to be inferior to method 2, because the natural criterion for choosing the $\{u_m\}$ was that they should be a good basis set for approximating y, which the delta functions certainly are not. A common fault of the Bateman kernel is that it contains spurious singularities. For example, if $K(t, s)$ has a discontinuity on the line $t = s$, then the Bateman kernel will normally have spurious discontinuities on the $2N$ lines $t = t_n$, $s = t_m$, $n, m = 1,\ldots, N$; the singularities at $t = t_n$ will appear in the approximate solution of the integral equation.

6. Numerical Examples

The first, and most thoroughly explored, of our two numerical examples is

$$y(t) = t^2 + \lambda \int_0^1 K(t, s)\, y(s)\, ds, \qquad 0 \leqslant t \leqslant 1, \tag{22}$$

where

$$K(t, s) = \begin{cases} t(2 - s), & t \leqslant s, \\ s(2 - t), & s \leqslant t, \end{cases}$$

which we have considered for a wide range of values of the parameter λ.

This is a convenient example, because the exact solution is known analytically from the solution of the corresponding differential equation

$$y''(t) + 2\lambda y(t) = 2,$$

with boundary conditions

$$y(0) = 0, \qquad y(1) + y'(1) = 3.$$

The characteristic values λ_n (i.e. the values of λ for which the homogeneous version of (22) has a nontrivial solution) are given by $\lambda_n = \mu_n^2/2$, $n = 1, 2,\ldots$, where μ_n is the nth positive root of $\tan \mu = -\mu$. The numerical values of the first few characteristic values are approximately

$$\lambda_1 = 2.05793, \qquad \lambda_2 = 12.06967,$$
$$\lambda_3 = 31.82955, \qquad \lambda_4 = 61.44458,\ldots .$$

The methods of Section 2, and also the method of moments, require a choice for the functions $\{u_n\}$, the principal criterion being that the functions should be a good basis set for approximating y. We learn from the integral equation itself that $y(t)$ necessarily vanishes at $t = 0$, and furthermore that y is a smooth function (in fact differentiable any number of times on (0, 1), with bounded derivatives). A convenient choice for the $\{u_n\}$ is therefore

$$u_n(t) = t^n, \qquad n = 1, 2,\ldots, N,$$

with which choice the subspace U_N is just the space of all polynomials of degree N that vanish at the origin. (An obvious advantage of the polynomial basis over, say, a trigonometric basis for this problem is that the inhomogeneous term in this problem is itself a polynomial, and therefore easily expressible in this basis. This is a point always worth considering in choosing the basis.) With this choice of

basis functions the various integrals that are needed (see Sect. 2) are easily evaluated analytically.

In Table I we give the $\lambda = 1$ results for methods 1 and 2 and the method of moments. Obviously, all of the approximate solutions are converging satisfactorily with increasing N, at each value of t. But the new methods are clearly much better than the method of moments, the errors being smaller by one or two factors of ten. The method 2 results are not shown for the largest value of N, because of some loss of significance through ill-conditioning. (Our general experience is that method 2 is less well conditioned than the other methods. Some ill-conditioning

TABLE I

Errors for First Example, with $\lambda = 1$

t	Exact solution, $y(t)$	Method[a]	$y(t) - y_N(t)$				
			$N = 2$	$N = 3$	$N = 4$	$N = 5$	$N = 6$
		M	0.	0.	0.	0.	0.
0.0	0.	1	0.	0.	0.	0.	0.
		2	0.	0.	0.	0.	0.
		M	−1.7(−2)	4.0(−4)	6.3(−6)	1.6(−6)	2.7(−7)
0.2	0.21505 05107	1	−6.9(−4)	6.2(−6)	−1.9(−7)	1.6(−8)	1.8(−9)
		2	−5.8(−4)	7.6(−6)	2.1(−7)	8.8(−9)	—
		M	−5.8(−3)	−4.2(−4)	−6.8(−5)	−9.5(−7)	−2.4(−7)
0.4	0.49247 94556	1	−2.0(−4)	−1.3(−5)	−1.2(−6)	−1.2(−8)	−2.1(−9)
		2	−1.6(−4)	−7.9(−6)	−7.6(−7)	−6.8(−9)	—
		M	1.2(−2)	−1.2(−4)	8.2(−5)	2.9(−8)	2.5(−7)
0.6	0.81024 00873	1	7.1(−4)	−6.6(−6)	1.4(−6)	−2.0(−9)	2.5(−9)
		2	4.8(−4)	−1.1(−6)	9.7(−7)	2.7(−9)	—
		M	1.2(−2)	4.3(−4)	−4.5(−5)	5.0(−7)	−2.7(−7)
0.8	1.14308 05761	1	7.7(−4)	7.3(−6)	−2.1(−7)	7.8(−9)	−1.4(−9)
		2	2.0(−4)	6.6(−6)	−7.1(−7)	−9.0(−10)	—
		M	−3.4(−2)	−8.5(−4)	2.5(−4)	4.2(−6)	−8.3(−7)
1.0	1.46455 07253	1	1.9(−4)	−3.6(−7)	6.2(−9)	−4.3(−12)	2.7(−10)
		2	−5.0(−4)	−6.9(−6)	8.8(−7)	1.0(−8)	—
		M	1.3(−2)	3.8(−4)	7.6(−5)	1.5(−6)	2.3(−7)
$\|y - y_N\|$		1	5.7(−4)	7.8(−6)	9.2(−7)	1.2(−8)	1.3(−9)
		2	3.9(−4)	6.1(−6)	6.8(−7)	8.9(−9)	—

[a] The method of moments is labeled by M, and methods 1 and 2 by 1 and 2 respectively. The numbers in parentheses are the powers of 10 by which the preceding numbers are to be multiplied.

problems should be expected in this example, since we have used a monomial basis, a basis set that is notorious in this respect.)

In the last line of the table we show also the L^2 error norm for the two methods,

$$\| y - y_N \| = (y - y_N, y - y_N)^{1/2}$$
$$= \left(\int_0^1 [y(t) - y_N(t)]^2 \, dt\right)^{1/2},$$

this being a convenient characterization of the average error over the interval. Obviously the values of $\| y - y_N \|$ are comparable in magnitude to the pointwise errors, and lead to the same qualitative conclusions. From now on we shall use this error norm to characterize the errors.

The behavior of the integral equation (1) can change greatly as λ is varied, and we have therefore thought it desirable to test the numerical methods over a range of different values of λ. In Fig. 1 we show the values of $\| y - y_N \|$ as a function of N for a number of positive values of λ, ranging from $\lambda = \frac{1}{4}$ (which is much smaller than the smallest characteristic value λ_1) to $\lambda = 64$ (which is

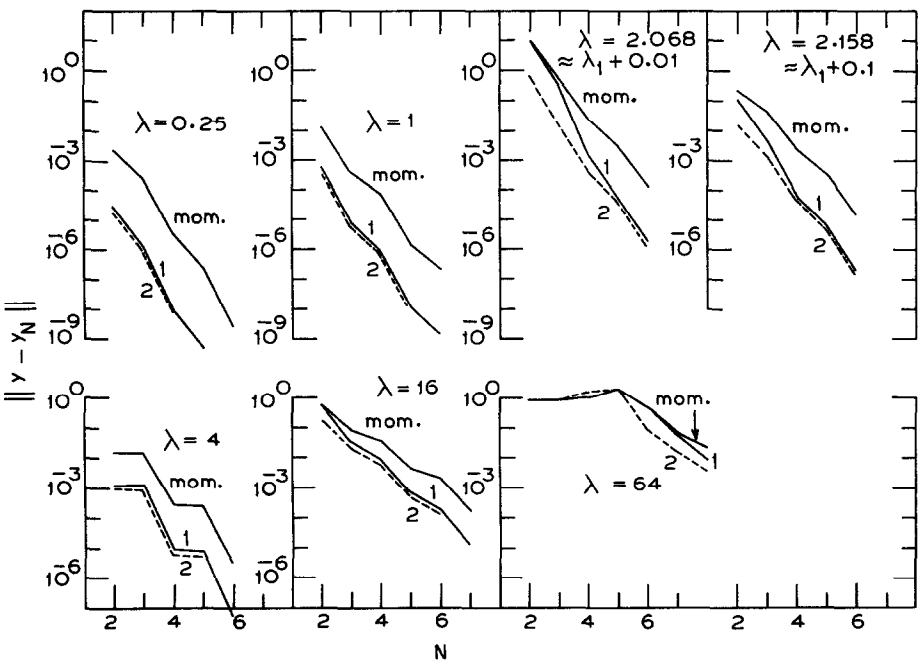

FIG. 1. Error norms $\| y - y_N \|$ for the first example, Eq. (22), for methods 1 and 2 and the method of moments, for $\lambda > 0$.

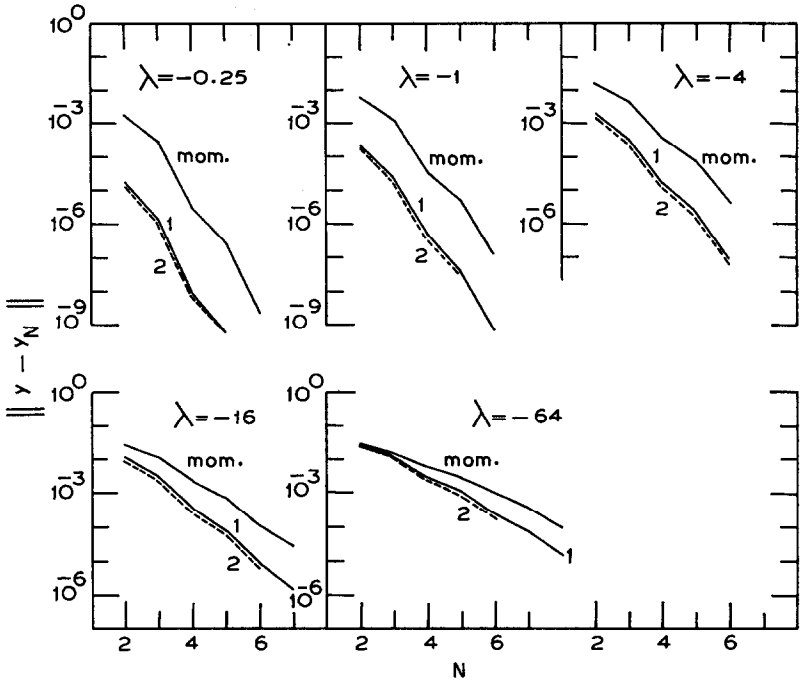

FIG. 2. Error norms $\| y - y_N \|$ for the first example, for $\lambda < 0$.

larger than λ_4). Two of the values are chosen to lie close to λ_1, to see what happens in that situation. Figure 2 shows similar results for several negative values of λ.

It is clear from these figures that all three methods behave satisfactorily as N increases, though the convergence becomes rather slow for large values of $|\lambda|$. (The slower convergence for large values of $|\lambda|$ is reasonably attributed to the fact that the polynomial basis set is much less appropriate in this case. In the worst case, $\lambda = 64$, the exact solution has four zeros in the interval [0, 1], and therefore is very poorly described by a polynomial of low degree.)

The most striking aspect of Figs. 1 and 2 is that the new methods developed in this paper virtually always give far smaller errors, by one or more powers of ten, than the method of moments; the only exceptions are where *all* the methods fail completely, such as for $\lambda = 64$ with small values of N. Particularly striking is the comparison between method 1 and the method of moments: these two methods are exactly equal in difficulty, yet evidently yield results of vastly different quality.

The more difficult of the new methods, method 2, is seen to give consistently better results than method 1, but the difference is often rather insignificant on

the logarithmic scales of Figs. 1 and 2. (In the regions where the curves are close, the ratio between the method 1 and 2 errors is about $\frac{3}{2}$.) However, for values of λ near the characteristic value λ_1, it is evident that method 2 is significantly more reliable than method 1.

The second of our numerical examples is the rather more difficult equation

$$y(t) = t^2 - \lambda \int_0^1 \ln | t - s | \, y(s) \, ds, \tag{23}$$

the kernel of which has logarithmic singularities on the line $t = s$. The exact solution is not known, but it is easily seen that $y(t)$ is not analytic at $t = 0$ and $t = 1$, and that $y'(t)$ is unbounded as $t \to 1$ from below. (Of course we could have contrived, in the usual way, to make the exact solution a known analytic function, but it seems to us more pertinent to find out if the methods work when the solution is *not* simple.)

The functions $\{u_n\}$ were again chosen to be polynomials,

$$u_n(t) = t^{n-1}, \qquad n = 1,..., N,$$

not because they are particularly appropriate to the problem (clearly they are not), but for convenience. In particular, the necessary integrals in Section 2 can then

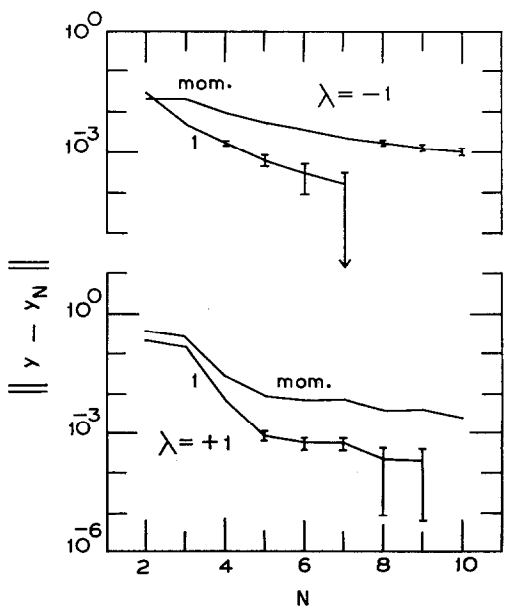

FIG. 3. Error norms $\| y - y_N \|$ for the second example, Eq. (23), for method 1 and the method of moments.

be evaluated analytically, by making use of the beta function integral and its derivatives.

The results for method 1 and the method of moments are shown in Fig. 3, for $\lambda = +1$ and $\lambda = -1$. (Method 2 was not studied for this example.) Obviously the convergence of both methods is much slower than in the first example, presumably because of the less satisfactory choice of basis functions.

The error bars in Fig. 3 arise from an uncertainty of 0.0002 in the "exact" values of $y(t)$. (The values of y used in Fig. 3 were obtained by increasing the values of N in method 1 until effective convergence was achieved. Then the quality of this result was tested by numerical integration of the right-hand side of (23), after suitable subtraction of the singularity at $t = s$, using Simpson's rule with 10, 20, 40,... points, again until effective convergence was achieved.)

It is strikingly clear from Fig. 3 that method 1 is again far superior to the method of moments. Except for small values of N, where *neither* method is useful, the errors in method 1 are smaller than those in the method of moments by more than a factor of ten, an improvement that could well make the practical difference between success and failure in an example of this type.

7. Conclusion

In both of the numerical examples, method 1 (the simpler of the two methods introduced in this paper) turns out to be far superior to the usual method of moments (Galerkin method)—a striking result, because the computational requirements are almost identical. A partial explanation is given in Section 3.

The theoretically preferred method 2 gives even better results in all cases where it has been studied, though the improvement over method 1 is often marginal.

References

1. F. Riesz and B. Sz.-Nagy, "Functional Analysis," Ungar, New York, 1955.
2. I. H. Sloan and S. K. Adhikari, *Nucl. Phys.* **A235** (1974), 352.
3. S. K. Adhikari and I. H. Sloan, Separable operator expansions for the t-matrix, *Nucl. Phys.*, to appear.
4. L. V. Kantorovich and V. I. Krylov, "Approximate Methods of Higher Analysis," Noordhoff, Groningen, 1958.
5. S. G. Mikhlin and K. L. Smolitskiy, "Approximate Methods for Solution of Differential and Integral Equations," Elsevier, New York, 1967.
6. C. T. H. Baker, "Proceedings of the Summer School on Integral Equations," Liverpool/Manchester 1973, Chap. 7, Oxford University Press, Oxford, 1974.
7. S. G. Mikhlin, "Variational Methods in Mathematical Physics," Pergamon, Oxford, 1964.
8. H. Bateman, *Proc. Roy. Soc. (London), Ser. A* **100** (1922), 441.